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INFLUENCE SURFACES FOR CONCENTRATED LOADS ON SLABS

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Introduction.

This paper is an attempt to determine the stresses in slabs due to concentrated loads. I hope to show that the most practical method of tackling the problem is by the use of influence surfaces corresponding to the influence lines used in the analysis of beams. The method of plotting the contours of, or sections through, these surfaces is discussed.

The thickness of a reinforced concrete slab is most often fixed by the permissible bending moment at the centre due to one or more concentrated loads such as the wheels of a road roller. In the case of roofs and floors a uniform load of which the effect is better known is more often the determining factor but even in this case a mental picture of the shape of the influence surface showing which parts of the load contribute most of the particular stress under consideration would be most useful.

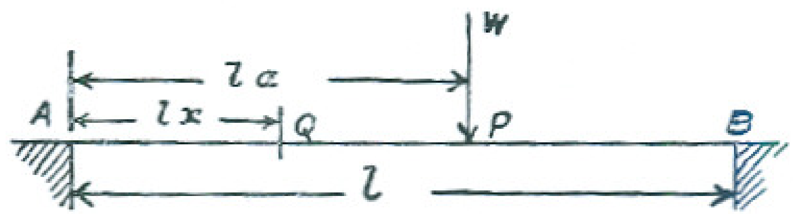
The present practice in dealing with concentrated loads is to consider the slab replaced by a beam of a definite width called the *effective width* and to make an allowance for the distribution of the load parallel to and at right angles to the span. The whole design, therefore, depends upon the formula for effective width and distribution. It is usual to consider the effective width as proportional to the span (0·6 of the span in one formula) and to add to this the width over which the load is distributed. This rule takes no account of the position of the load; it assumes an influence surface whose section parallel to the shorter span is constant and the same as the influence line of a beam. If this assumption were correct no allowance should be made for distribution of the load at right angles to the span as such a load would have the same effect as a concentrated one. I propose to show that the influence surface for bending moment at the centre of a slab bears no resemblance whatsoever to a surface whose cross section is everywhere the influence line of a beam. The one has the shape of the horn of an old-fashioned gramophone; the other that of a triangular ruler.

'Effective Width' must therefore be abandoned; in its place I propose to substitute a number of contoured surfaces for quantities, such as the deflection and bending moment at the centre. The deflection and bending moment due to any particular load will then be found by multiplying the weight of the load by the height of the influence surface at the point where it is placed. Should the load be distributed it may be divided up into sections and each section treated as concentrated or the whole load may be multiplied by the average height of the surface over the same area as that occupied by it.

The formula for the influence surface of a bending moment for instance will give the bending moment due to a unit load for a unit span as a function of the position of the load (α, β), and the position of the section (x, y) where the bending moment is required.

In order to illustrate the method used I propose to apply it in the first place to the problem of finding influence lines of beams.

Theory—Beam Freely Supported.



Notation.

l—	span.
lx—	position of section—Q.
$l\alpha$—	,, ,, load—P.
lz—	deflection.
w—	load per unit length at P.
W—$wld\alpha$—	load on element of length at P.
u—	influence line of deflection
i—	,, ,, ,, slope.
m—	,, ,, ,, bending moment.
s—	,, ,, ,, shear.

From the influence lines the deflection, slopes, etc., at x are found from the following formulae ; the first is for a concentrated load, the second for a uniform load :

	Concentrated.	*Uniform.*
Deflection at x	$\frac{Wl^3}{EI}u$	$\frac{wl^4}{EI}\int_0^1 u d\alpha$
Slope—— ,,	$\frac{Wl^2}{EI}i$	$\frac{wl^3}{EI}\int_0^1 u d\alpha$
Bending moment ,,	$Wl\, m$	$wl^2\int_0^1 m d\alpha$
Shear—— ,,	Ws	$wl\int_0^1 s d\alpha$

It should be noted that x, α, z, u, i, m and s are pure numbers of no dimensions and are therefore unaffected by changes in the units of length and weight. This accounts for factors such as l^3 in equation (I) which are not met with when x, z have the dimensions of length.

We start with the well known equation giving the deflection at a point as a function of the load at that point :—

$$w' = \frac{EI}{l^3}\frac{d^4z}{dx^4}$$

w having already been defined as the load at P the accent is used for a load elsewhere. Here w' is supposed to vary over the whole span. Let us assume it to take the form of a concentrated load $wld\alpha$ at P and divide by w.

$$\frac{w'}{w} = \frac{EI}{wl^3}\frac{d^4z}{dx^4} \qquad (2)$$

The expression on the left of the equation has the following form : 0 form A to P ; 1 over the interval $d\alpha$ at P and 0 from P to B. It can be expressed as a half-range Fourier series :

$$\frac{w'}{w} = d\alpha\, \Sigma\, a_n \sin n\pi x \qquad (3)$$

$$\text{where } a_n = 2\int_0^1 \frac{w'}{wd\alpha} \sin n\pi x d x$$

$$= 2 \sin n\pi\alpha \qquad (4)$$

Substituting in equation (3) the expression on the right (2) we can solve the differential equation for z.

$$u = \frac{EI}{wl^3 d\alpha} z = \frac{1}{\pi^4}\Sigma\frac{a_n}{n^4} \sin n\pi x \qquad (5)$$

This equation gives the influence line for deflection as an infinite series. The other influence lines are got directly by differentiating :—

$$\text{Deflection——— } u = \frac{1}{\pi^4}\Sigma\frac{a_n}{n^4} \sin n\pi x \qquad (6)$$

$$\text{Slope——— } i = \frac{du}{dx} = \frac{1}{\pi^3}\Sigma\frac{a_n}{n^3} \cos n\pi x \qquad (7)$$

$$\text{Bending moment } m = (-)\frac{d^2u}{dx^2} = \frac{1}{\pi^2}\Sigma\frac{a}{n^2} \text{snn}\, \pi x \qquad (8)$$

$$\text{Shear—— } s = (-)\frac{d^3u}{dx^3} = \frac{1}{\pi}\Sigma\frac{a_n}{n} \cos n\pi x \qquad (9)$$

In the case of a uniform load we must integrate a_n in equation (4) with respect to α over the whole span.

$$\int_0^4 a_n d\alpha = \frac{4}{n\pi} \qquad (10)$$

In making use of these formulæ the following summations should be remembered :—

$$\left.\begin{aligned}
1-\frac{1}{3}+\frac{1}{5}-\ \ldots\ldots\ldots\ldots &= \frac{\pi}{4}\\
1+\frac{1}{3^2}+\frac{1}{5^2}+\ \ldots\ldots\ldots\ldots &= \frac{\pi^2}{8}\\
1-\frac{1}{3^3}+\frac{1}{5^3}-\ \ldots\ldots\ldots\ldots &= \frac{\pi^3}{32}\\
1+\frac{1}{3^4}+\frac{1}{5^4}+\ \ldots\ldots\ldots\ldots &= \frac{\pi^4}{96}\\
1-\frac{1}{3^5}+\frac{1}{5^5}-\ \ldots\ldots\ldots\ldots &= \frac{5}{1536}\pi^5\\
1+\frac{1}{2^2}+\frac{1}{3^2}+\ \ldots\ldots\ldots\ldots &= \frac{\pi^2}{6}\\
1-\frac{1}{2^2}+\frac{1}{3^2}-\ \ldots\ldots\ldots\ldots &= \frac{\pi^2}{12}
\end{aligned}\right\}\qquad(11)$$

Examples.

Deflection at centre.—Concentrated load at centre.

From equations (4) and (6) :

$$u=\frac{2}{\pi^4}\Sigma\frac{1}{n^4}\ \ldots\ldots\ldots\ldots\ (n\text{ odd})$$

$$=\frac{2}{\pi^4}\times\frac{\pi^4}{96}=\frac{1}{48}$$

$$\therefore\ \text{deflection}=\frac{1}{48}\frac{Wl^3}{EI}\qquad(12)$$

Deflection at centre.—Uniform load.

From equations (6) and (10):

$$\int_0^1 uda=\frac{4}{\pi^5}\Sigma\frac{1}{n^5}\ \ldots\ldots\ldots\ (n\text{ odd alternating})$$

$$=\frac{4}{\pi^5}\times\frac{5}{1536}\pi^5=\frac{5}{384}$$

$$\therefore\ \text{deflection}=\frac{5}{384}\frac{wl^4}{EI}\qquad(13)$$

Bending moment at centre.—Concentrated load at centre. $x=a=\frac{1}{2}$,

From equations (4) and (8) :

$$m=\frac{2}{\pi^2}\Sigma\frac{1}{n^2} \quad \ldots\ldots\ldots\ldots \quad (n \text{ odd})$$

$$=\frac{2}{\pi^2}\times\frac{\pi^2}{8}=\frac{1}{4}$$

$$\therefore \text{ bending moment}=\frac{Wl}{4} \tag{14}$$

Bending moment at centre.—Uniform load.

From equations (8) and (10)

$$m=\frac{4}{\pi^3}\Sigma\frac{1}{n^3}\ldots\ldots(n \text{ odd alternating})$$

$$=\frac{4}{\pi^3}\times\frac{\pi^3}{32}=\frac{1}{8}$$

$$\therefore \text{ bending moment}=\frac{wl^2}{8} \tag{15}$$

Theory—Beam Fixed.

In the case of a fixed beam the half-range cosine series gives :—

$$\frac{d^4u}{dx^4}=1+\Sigma b_n \cos n\pi x \tag{16}$$

$$\text{where } b_n=2\cos n\pi a \tag{17}$$

$$\text{and } \int_0^1 b_n\,da=0 \tag{18}$$

The solution of (16) gives the influence lines for deflection and bending moment :—

$$u=\frac{x^4}{24}-\frac{x^3}{12}+\frac{x^2}{24}+\frac{1}{\pi^4}\Sigma\frac{a_n}{n^4}\left(\cos n\pi x-1\right)$$
$$-\frac{a_{2n-1}}{(2n-1)^4}\left(4x^3-6x^2\right) \tag{19}$$

$$m=-\frac{x^2}{2}+\frac{x}{2}-\frac{1}{12}+\frac{1}{\pi^2}\Sigma\frac{a_n}{n^2}\cos n\pi x$$
$$+\frac{a_{2n-1}}{\pi^2(2n-1)^4}\left(24x-12\right) \tag{20}$$

With a uniform load as a result of equation (18) only the first three terms are required. The problem of the fixed beam was tackled in the hope that the solution might help in the analogous problem of the fixed slab but this I have not been able to solve.

Theory—Slab supported along four edges.

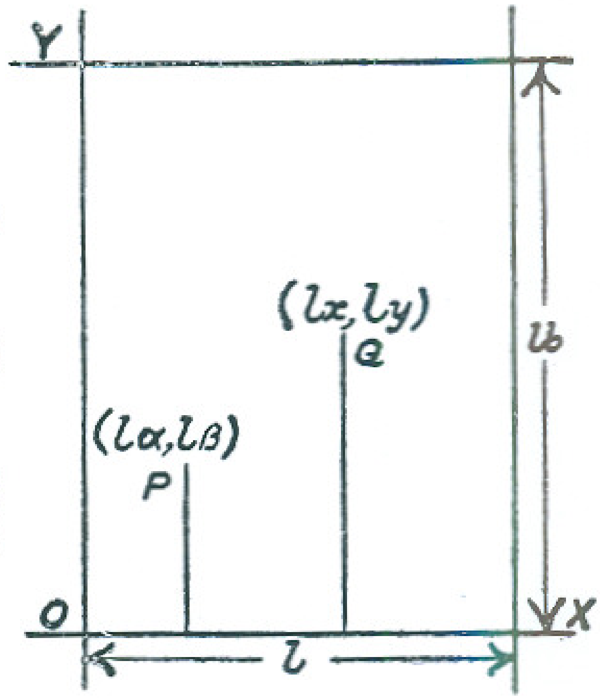
Notation.

l—shorter span.

lb—longer span or the length for maximum deflection.

α, β—co-ordinates of position of load for unit span—P.

x, y— do. of point at which the bending moment, etc., is calculated—Q.



zl—deflection.

w—load per unit area at P.

W— $wl^2\,d\,\alpha\,d\,\beta$—load on element at P,

σ—Poisson's ratio$=\frac{1}{4}$.

M_1—bending moment per unit length on a section parallel to longer side.

$$E' = \frac{E}{1-\sigma^2} = \frac{16}{15}\,E,$$

u—height of influence surface for deflection.

m_1— do. for bending moment M_1.

Deflection and Bending Moment are obtained from the heights of the influence surfaces by the following formulae for a concentrated load and for a uniform load.

	Concentrated.	*Uniform.*
Deflection	$\frac{Wl^2}{E'I}\,u$	$\frac{wl^4}{E'I}\iint u\,d\,\alpha\,d\,\beta.$
Bending moment	$W\,m_1$	$wl^2\iint m_1\,d\,\alpha\,d\,\beta.$

As the purpose of this paper is rather to introduce a method and to discuss the question of effective width, than to give tabulated results the influence surface of the principal bending moment only is discussed. It should, however, be remembered that corresponding to every point of the slab there are : one influence surface for deflection, two for slopes, two for bending moments and two for shears.

Corresponding to equation (1) we have for the slab the equation :

$$w' = \frac{E'I}{l^3}\left(\frac{\delta^4 z}{\delta x^4} + 2\frac{\delta^4 z}{\delta x^2 \delta y^2} + \frac{\delta^4 z}{\delta y^4}\right) \qquad (21)$$

(See Applied Elasticity by Prescott, p. 391, with the notation slightly altered).

This equation gives the deflection z (for unit span) as a function of w' the load per unit area at the same point. We can consider any distribution of w' we please. Let us replace it by a concentrated load on the area $d\alpha \; d\beta$ of intensity w and zero over the rest of the slab. This function can be expressed as a two-dimensional Fourier series :

$$\frac{w'}{w} = d\alpha \, d\beta \, \Sigma \, A_{mn} \sin m\pi x \, \sin n\pi \frac{y}{b} \qquad (22)$$

This is to be summed for all values of m and n and the A_{mn} are to be chosen so that the function which represents a surface takes the shape of the load distribution corresponding to a concentrated load. This is a lot to ask of a continuous function and we must be prepared for failures.

The values of A_{mn} are :—

$$A_{mn} = \frac{4}{b\, d\alpha\, d\beta}\int_0^1\int_0^b \frac{w'}{w} \sin m\pi x \sin n\pi \frac{y}{b}$$

$$= \frac{4}{b} \sin m\pi\alpha \sin n\pi \frac{\beta}{b} \qquad (23)$$

For a uniform load integrate with respect to α and β over the whole surface :—

$$\int_0^1\int_0^b A_{mn}\, d\alpha\, d\beta = \frac{16}{\pi^2 m n} \quad (m, n \text{ odd}) \qquad (24)$$

Combining equations (21) and (22) the deflection due to a concentrated load can be expressed as a Fourier series :

$$\frac{E'I}{wl^3}\left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}\right)^2 z = d\alpha d\beta \, \Sigma A_{mn} \sin m\pi x \sin n\pi \frac{y}{b} \qquad (25)$$

This equation has many solutions of which that which will satisfy the boundary conditions of the particular problem is required. In the case of a slab supported on all sides the conditions are that the deflection and bending moment shall be zero when $x=0, 1$ and $y=0, b$. This is satisfied by the solution :—

$$u=\frac{E'I}{wl^3\, d\,a\,d\,\beta}\,z=\frac{1}{\pi^4}\,\Sigma\,\frac{A_{mn}}{\left(m^2+\frac{n^2}{b^2}\right)^2}\,\sin m\,\pi\,x\,\sin n\,\pi\,\frac{b}{y} \quad (26)$$

which gives the influence surface of deflection. Differentiating this partially with respect to x and y we get the influence surface for the principal bending moment.

$$m_1=-\frac{EI'}{wl^3\,d\alpha\,d\beta}\left(\frac{\delta^2}{\delta x^2}+\sigma\,\frac{\delta^2}{\delta y^2}\right)z$$

$$=\frac{1}{\pi^2}\,\Sigma\,\frac{bm^2+\sigma n^2}{\left(bm^2+\frac{n^2}{b}\right)^2}\,A_{mn}\,\sin m\,\pi\,x\,\sin n\,\pi\,\frac{y}{b} \qquad (27)$$

Deflection at centre.

Consider the deflection at centre due to a load at the centre. In his case :

$x=\frac{y}{b}=\alpha=\frac{\beta}{b}=\frac{1}{2}$ and equations (23) and (26) give :

$$u=\frac{4}{b\pi^4}\,\Sigma\,\frac{1}{\left(m^2+\frac{n^2}{b^2}\right)^2} \quad (m,\ n\ \text{odd}) \qquad (28)$$

Now b has been defined as the ratio between the sides and if we consider it infinite this eqbation shows that u will be zero, which is absurd. The mistake lies in equation (26) which assumes that there will be no corrugations between supports. Now a load in the centre will take up the position of minimum work which corresponds to maximum deflection and as it cannot uniformly deflect an infinitely long slab even slightly, it will cause corrugations and the problem of the long slab is solved by finding for which value of b the deflection is greatest :

From equation (28) I have calculated the values of u for one or two different values of b :

b =	1	2	3	4	10
n =	0·0116	0·0164	0·0168	0·0168	0·0165

This shows that the greatest deflection occurs when b is 3 or 4, and in treating an infinitely long slab I shall therefore adopt the value 4. We shall expect all slabs whose length is more than twice their width to behave in the same way where the loading occupies a small area. It is interesting to see what the width of a beam must be to deflect the same amount as a long slab. Let this width be h.

$$\frac{wl^2}{E'I}\,0{\cdot}0168 = \frac{1}{48}\,\frac{wl^3}{hEI}$$

$$\therefore\ h = 1{\cdot}25\,l$$

That is, the effective width in this particular case is 1·25 times the span.

Bending moment at centre.

For a slab which is not continuous the most useful influence surface is that of the bending moment of the centre of a long slab.

In this case $x=\dfrac{y}{b}=\frac{1}{2}$, $b=4$, $\sigma=\frac{1}{4}$ and equation (27) becomes :

$$m_1 = \frac{4}{\pi^2}\,\Sigma\,\frac{4m^2+n^2}{\left(4m^2+\dfrac{n^2}{4}\right)^2}\,\sin m\,\pi\,\alpha\,\sin n\,\pi\,\frac{\beta}{4} \qquad (29)$$

m and n are to be summed for odd numbers and the terms in the double series will be alternately positive and negative for both m and n sr the equation may be multiplied by $(-)^{\frac{m+n-2}{2}}$ The influence surface is plotted by taking different values of α and β and calculating the corresponding value of m_1,—a very laborious process but one which I have not been able to simplify. If we write

$$S_m = \Sigma\,\frac{4m^2+n^2}{\left(4m^2+\dfrac{n^2}{4}\right)^2} \text{ summed for all positive odd values of } n$$

and let S_n represent the same expression summed for all positive odd values of m, values of S_m and S_n will be found very useful for points on the axes of the slab.

They are given in the following table :

m or n ..	1	3	5	7	9	11	13
S_m ..	2·272	0·656	0·394	0·283	0·221	0·183	0·155
S_n ..	0·335	0·395	0·342	0·271	0·216	0·178	0·151
m or n ..	15	17	19	21	23	25	27
S_m ..	0·134	0·118	0·106	0·097	0·089	0·082	0·076
S_n ..	0·131	0·115	0·104	0·095	0·088	0·081	0·075
m or n ..	29	31	33	35	37	39	41
S_m ..	0·070	0·065	0·061	0·057	0·053	0·050	0·048
S_n ..	0·070	0·065	0·061	0·057	0·053	0·050	0·048

It simplifies the calculation if the half span is divided into six equal parts.

Load at centre.

For a load at the centre $\alpha = \frac{\beta}{4} = \frac{1}{2}$ and equation (29) becomes :—

$$m_1 = \frac{4}{\pi^2} \Sigma \frac{4m^2+n^2}{\left(4m^2+\frac{n^2}{4}\right)^2} \qquad (30)$$

to be summed for all positive odd values of m and n. Dr. Chowla of Delhi has very kindly shown me that this series is divergent which means that the bending moment directly under a concentrated load is infinite. In practice, however, there is no such thing as a load concentrated at a point and the surface can be rounded off without affecting its usefulness provided the volume is kept constant.

Fig. 1 shows contours of the influence surface, Fig 2. sections parallel to the shorter span and Fig. 3 sections perpendicular to that span. It will be noticed that a load anywhere in a square of side equal to the span l will cause a positive moment at the centre of the square. Outside that area a load will give a very slight negative moment at the same point. The effective width for a load at any point for which α is not greater than $\frac{1}{2}$ is equal to $\frac{\alpha l}{2m_1}$. The contours show that this is a very variable quantity. Two lines have been drawn for which the effective width is equal to the span. Inside these lines i: is less, being zero at the centre, and outside it is greater than the span.

The effect of distribution is seen by comparing the average height of the surface over the area occupied by the load with the height at the centre of that area. At the centre of the slab the effect of distributing the load is enormous while at some parts it is almost negligible.

Uniform load.

For a load over the whole slab equations (24) and (27) give :

$$\int_0^1 \int_0^4 m_1 \, d\,\alpha \, d\,\beta = \frac{64}{\pi^4} \Sigma \frac{4m^2+n^2}{mn\left(4m^2+\frac{n^2}{4}\right)^2} \tag{31}$$

$$= 0.128 = \tfrac{1}{8}$$

This is the algebraic volume of the influence surface and the corresponding bending moment is of course the same as for a beam freely supported.

The greatest positive moment will be got by loading only the centre square and the integration with respect to β must be between the limits 1.5 and 2.5. The right side of equation (24) then becomes :

$$\frac{16}{\pi^2 mn} \cos \tfrac{3}{8} n \pi \tag{32}$$

Multiplying equation (31) by $\cos \frac{3}{8} n \pi$ we get for the volume of the influence surface above the xy plane the value 0.159 and the corresponding bending moment 0.159 wl^2. The effect of a line load along the centre perpendicular to the span is got by integrating with respect to β only. The area of the section on one side of the centre is shown in Fig 3 ; it has been plotted from the equation

$$\int_c^2 m_1 \, d\,\beta = \frac{16}{\pi^3} \Sigma \frac{4m^2+n^2}{n\left(4m^2+\frac{n^2}{4}\right)^2} \cos n \pi \frac{c}{4} \tag{33}$$

(alternating)

When $c=o$ this gives half the area of the section. The area of the whole curve is $\frac{1}{4}$ and the corresponding bending moment is $\frac{wl}{4}$ which is that resulting from a load in the centre of a b:am.

Fig. 4 shows the contours of the influence surface for bending moment at the centre of a square slab supported along its four edges. The volume in this case is 0.046 and the corresponding bending moment 0.046 wl^2.

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Instructions for preparation of diagrams to accompany papers.

All drawings should be submitted the actual size for reproduction, the limit of size is $7\frac{1}{2}$ inches in height and 15 inches in length. These limits should on no account be exceeded.

All drawings should be in one colour only. In addition to a complete drawing a tracing on the mat side (not the glossy side) of the cloth but without any lettering or printing should also be sent. The lettering will be arranged by the Honorary Secretary.

These tracings should be in good black ink, Pelican Waterproof ink is recommended, and should be rolled, not folded.

The salient points of a diagram are more easily appreciated if there is little printing on it. Accordingly the explanations should be given in the text and not on the diagram and all dimensions that are not essential should be omitted.

The number of plates of drawings to be submitted with each paper should not exceed six, unless the Council agree to allow an increase in any special case.